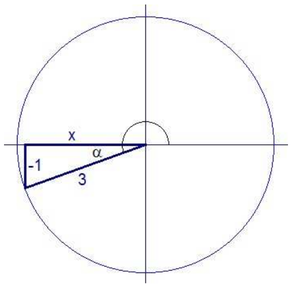
Ejercicio 1.

Si $cosec\,\alpha=-3$, $\pi<\alpha<\frac{3\pi}{2}$, determina, sin usar la calculadora, el valor de $cotg\frac{\alpha}{2}$, $tg\,2\alpha$, $cos(\pi-2\alpha)$ y $sen\left(\frac{\pi}{2}+\alpha\right)$.

$cosec\,\alpha=-3$ y $\pi<\alpha<\frac{3\pi}{2}$ $\Rightarrow$ *como vemos en la figura, podemos calcular las razones trigonométricas del ángulo* α *sobre el triángulo marcado, donde* $x=-2\sqrt{2}$

entonces tenemos que $sen\,\alpha=-\frac{1}{3}$, $cos\,\alpha=-\frac{2\sqrt{2}}{3}$, $tg\,\alpha=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$



$$tg\,2\alpha=\frac{sen\,2\alpha}{cos\,2\alpha}=\frac{2\cdot sen\,\alpha\cdot cos\,\alpha}{cos^2\alpha-sen^2\alpha}=\frac{2\cdot\left(-\frac{1}{3}\right)\cdot\left(-\frac{2\sqrt{2}}{3}\right)}{\frac{8}{9}-\frac{1}{9}}=\frac{\frac{4\sqrt{2}}{9}}{\frac{7}{9}}=\frac{4\sqrt{2}}{7}$$

$$cotg\frac{\alpha}{2}=\frac{cos\frac{\alpha}{2}}{sen\frac{\alpha}{2}}\overset{(1)}{=}-\frac{\sqrt{1+cos\,\alpha}}{\sqrt{1-cos\,\alpha}}=-\sqrt{\frac{1-\frac{2\sqrt{2}}{3}}{1+\frac{2\sqrt{2}}{3}}}=-\sqrt{\frac{3-2\sqrt{2}}{3+2\sqrt{2}}}=-\sqrt{\frac{\left(3-2\sqrt{2}\right)^2}{\left(3+2\sqrt{2}\right)\left(3-2\sqrt{2}\right)}}=-\sqrt{\frac{17-12\sqrt{2}}{1}}=-\sqrt{17-12\sqrt{2}}$$

$$cos(\pi-2\alpha)=-cos\,2\alpha=-\left(cos^2\alpha-sen^2\alpha\right)=-\left(\frac{8}{9}-\frac{1}{9}\right)=-\frac{7}{9}$$

$$sen\left(\frac{\pi}{2}+\alpha\right)=sen\frac{\pi}{2}\cdot cos\,\alpha+cos\frac{\pi}{2}\cdot sen\,\alpha=cos\,\alpha=-\frac{2\sqrt{2}}{3}$$

$$(1)\left(si\ \ \pi<\alpha<\frac{3\pi}{2}\ \Rightarrow\ \frac{\pi}{2}<\frac{\alpha}{2}<\frac{3\pi}{4}\ \Rightarrow\ cotg\frac{\alpha}{2}<0\right)$$

Ejercicio 2.

Resuelve la ecuación trigonométrica: $sen\,2x+cos\,2x+1=sen\,x+cos\,x$

$$sen\,2x+cos\,2x+1=sen\,x+cos\,x\ \Rightarrow\ 2sen\,x\cdot cos\,x+cos^2x-sen^2\,x+1=sen\,x+cos\,x\ \Rightarrow$$

$$\Rightarrow\ 2sen\,x\cdot cos\,x+cos^2x-\left(1-cos^2x\right)+1=sen\,x+cos\,x\ \Rightarrow\ 2sen\,x\cdot cos\,x+2cos^2x=sen\,x+cos\,x\ \Rightarrow$$

$$\Rightarrow\ 2cos\,x\cdot(cos\,x+sen\,x)=sen\,x+cos\,x\ \Rightarrow\ 2cos\,x\cdot(cos\,x+sen\,x)-(sen\,x+cos\,x)=0\ \Rightarrow$$

$$\Rightarrow\ (2cos\,x-1)\cdot(cos\,x+sen\,x)=0\ \Rightarrow$$

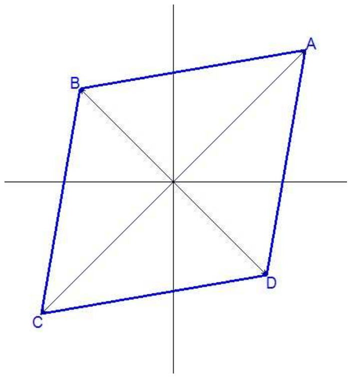
$$\Rightarrow\begin{cases}2cos\,x-1=0\ \Rightarrow\ cos\,x=\frac{1}{2}\ \Rightarrow\ \begin{cases}x=60^\circ+k\cdot360^\circ\\x=300^\circ+k\cdot360^\circ\end{cases}\\ cos\,x+sen\,x=0\ \Rightarrow\ cos\,x=-sen\,x\ \Rightarrow\ \begin{cases}x=135^\circ+k\cdot360^\circ\\x=315^\circ+k\cdot360^\circ\end{cases}\end{cases}$$

Ejercicio 3.

Las raíces de la ecuación $x^4 - 2i \cdot x^2 + 8 = 0$ son vértices de una figura plana. Hállense el área y el perímetro de dicha figura.

$$x^4 - 2ix^2 + 8 = 0 \quad \left(\textit{hacemos el cambios de variable } x^2 = y\right) \quad \Rightarrow \quad y^2 - 2iy + 8 = 0$$

$$y = \frac{2i \pm \sqrt{(-2i)^2 - 4\cdot 8}}{2} = \frac{2i \pm \sqrt{-36}}{2} = \frac{2i \pm \sqrt{36}\cdot\sqrt{-1}}{2} = \frac{2i \pm 6i}{2} \Rightarrow \begin{cases} y = 4i \\ y = -2i \end{cases}$$



$$x^2 = 4i \Rightarrow x = \sqrt{4i} \Rightarrow x = \sqrt{4_{90^\circ}} \Rightarrow \begin{cases} x_1 = \sqrt{4}_{\frac{90^\circ}{2}} & \Rightarrow x_1 = 2_{45^\circ} \\ x_2 = \sqrt{4}_{\frac{90^\circ + 360^\circ}{2}} & \Rightarrow x_2 = 2_{225^\circ} \end{cases}$$

$$x^2 = -2i \Rightarrow x = \sqrt{-2i} \Rightarrow x = \sqrt{2_{270^\circ}} \Rightarrow \begin{cases} x_3 = \sqrt{2}_{\frac{270^\circ}{2}} & \Rightarrow x_3 = \sqrt{2}_{135^\circ} \\ x_4 = \sqrt{2}_{\frac{270^\circ + 360^\circ}{2}} & \Rightarrow x_4 = \sqrt{2}_{315^\circ} \end{cases}$$

Las raíces de la ecuación son los vértices de un rombo:

$A = 2_{45^\circ}\ ,\ B = \sqrt{2}_{135^\circ}\ ,\ C = 2_{225^\circ}\ ,\ D = \sqrt{2}_{315^\circ}$

$$\left.\begin{array}{l} \textit{Las diagonales miden } 4 \textit{ y } 2\sqrt{2} \\ \textit{El lado mide } \sqrt{6} \end{array}\right\} \Rightarrow \left| \begin{array}{l} \textit{Área} = \dfrac{4\cdot 2\sqrt{2}}{2} = 4\sqrt{2}\ u^2. \\ \textit{Perímetro} = 4\sqrt{6}\ u. \end{array}\right.$$

Ejercicio 4.

Dado el complejo $z = \left(-2\sqrt{3}\,,\,2\right)$, se pide:

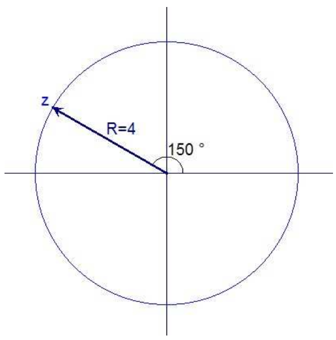
a) Escríbelo en todas las formas posibles y nómbralas.

b) Halla z^6 y expresa el resultado en forma binómica.

c) Halla las raíces cúbicas del número $z \cdot i^{27}$ y represéntalas en el plano.

$z = \left(-2\sqrt{3}, 2\right) \rightarrow$ *forma cartesiana del número complejo*

$z = -2\sqrt{3} + 2i \rightarrow$ *forma binómica*



$$z = -2\sqrt{3} + 2i \Rightarrow \begin{cases} R = \sqrt{\left(-2\sqrt{3}\right)^2 + 2^2} = \sqrt{16} = 4 \\ \alpha = arctg\left(-\dfrac{2}{2\sqrt{3}}\right) = 150^\circ,\ \textit{al ser}\ 90^\circ < \alpha < 180^\circ \end{cases}$$

$z = 4_{150^\circ} \rightarrow$ *forma polar*

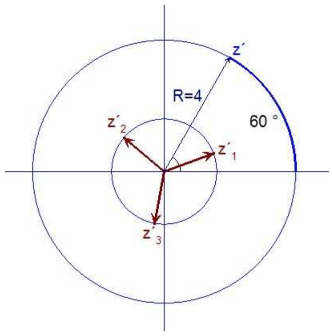
$z = 4\cdot\left(cos\,150^\circ + i\,sen\,150^\circ\right) \rightarrow$ *forma trigonométrica.*

$$z^6 = \left(4_{150^\circ}\right)^6 \Rightarrow z^6 = 4^6{}_{6\cdot 150^\circ} \Rightarrow z^6 = 4096_{900^\circ} \Rightarrow z^6 = 4096_{180^\circ} \Rightarrow z^6 = -4096 + 0\cdot i$$

$$z' = z \cdot i^{27} \;\Rightarrow\; z' = z \cdot i^{3} \;\Rightarrow\; z' = \left(-2\sqrt{3} + 2i\right)\cdot(-i) \;\Rightarrow\; z' = -2i^2 + 2\sqrt{3}\,i \;\Rightarrow\; z' = 2 + 2\sqrt{3}\,i$$

$$z' = 2 + 2\sqrt{3}\,i \;\Rightarrow\; \begin{cases} R = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{16} = 4 \\ \beta = arctg\left(\dfrac{2\sqrt{3}}{2}\right) = 60^\circ, \;\; al\ ser\ \ 0^\circ < \beta < 90^\circ \end{cases} \;\Rightarrow\; z' = 4_{60^\circ}$$

$$\sqrt[3]{z'} = \sqrt[3]{4_{60^\circ}} \;\Rightarrow\; \begin{cases} z_1' = \sqrt[3]{4}_{\frac{60^\circ}{3}} & \Rightarrow\; z_1' = \sqrt[3]{4}_{20^\circ} \\ z_2' = \sqrt[3]{4}_{\frac{60^\circ+360^\circ}{3}} & \Rightarrow\; z_2' = \sqrt[3]{4}_{140^\circ} \\ z_3' = \sqrt[3]{4}_{\frac{60^\circ+2\cdot 360^\circ}{3}} & \Rightarrow\; z_3' = \sqrt[3]{4}_{260^\circ} \end{cases}$$



Ejercicio 5.

Determina el valor exacto de las razones trigonométricas de los ángulos de $15^\circ,\ 75^\circ,\ 165^\circ,\ 255^\circ$.

$$sen\,15^\circ = sen\left(45^\circ - 30^\circ\right) = sen\,45^\circ \cdot cos\,30^\circ - sen\,30^\circ \cdot cos\,45^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$cos\,15^\circ = cos\left(45^\circ - 30^\circ\right) = cos\,45^\circ \cdot cos\,30^\circ + sen\,30^\circ \cdot sen\,45^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$tg\,15^\circ = \frac{sen\,15^\circ}{cos\,15^\circ} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\left(\sqrt{6}-\sqrt{2}\right)^2}{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)} = \frac{8-4\sqrt{3}}{4} = 2-\sqrt{3}$$

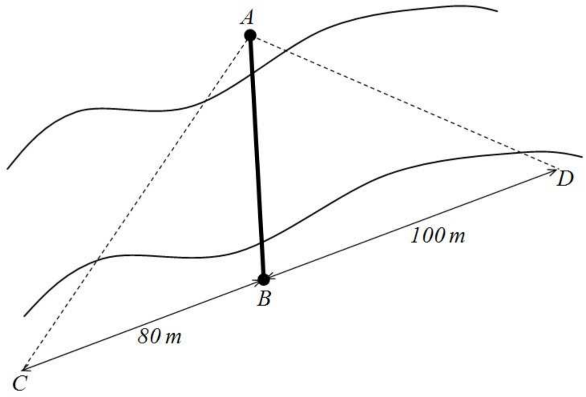
$$\boxed{75^\circ = 90^\circ - 15^\circ} \;\Rightarrow\; \begin{cases} sen\,75^\circ = cos\,15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4} \\ cos\,75^\circ = sen\,15^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4} \\ tg\,75^\circ = cotg\,15^\circ = \dfrac{1}{2-\sqrt{3}} = 2+\sqrt{3} \end{cases}$$

$$\boxed{165^\circ = 180^\circ - 15^\circ} \;\Rightarrow\; \begin{cases} sen\,165^\circ = sen\,15^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4} \\ cos\,165^\circ = -cos\,15^\circ = -\dfrac{\sqrt{6}+\sqrt{2}}{4} \\ tg\,165^\circ = -tg\,15^\circ = \sqrt{3}-2 \end{cases}$$

$$\boxed{255^\circ = 180^\circ + 75^\circ} \;\Rightarrow\; \begin{cases} sen\,255^\circ = -sen\,75^\circ = -\dfrac{\sqrt{6}+\sqrt{2}}{4} \\ cos\,255^\circ = -cos\,75^\circ = \dfrac{\sqrt{2}-\sqrt{6}}{4} \\ tg\,255^\circ = tg\,75^\circ = 2-\sqrt{3} \end{cases}$$

Ejercicio 6.

Se quiere construir un puente sobre un río que una los puntos *A* y *B*. Para ello elegimos dos puntos *C* y *D*, a ambos lados del punto *B* y alineados con él, y medimos los ángulos $B\hat{C}A = 30^{\circ}$, $B\hat{D}A = 45^{\circ}$. Determina la longitud del puente sabiendo que $\overline{BC} = 80\ m$ y $\overline{BD} = 100\,m$.



Aplicamos el teorema de los senos en el triángulo ACD.

Tenemos que el ángulo $C\hat{A}D = 180^{\circ} - 30^{\circ} - 45^{\circ} = 105^{\circ}$

$$\frac{180}{sen\,105^{\circ}} = \frac{\overline{AD}}{sen\,30^{\circ}} \quad \Rightarrow \quad \overline{AD} = \frac{180 \cdot sen\,30^{\circ}}{sen\,105^{\circ}} = \frac{180 \cdot \frac{1}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{360}{\sqrt{6}+\sqrt{2}} = 90\left(\sqrt{6}-\sqrt{2}\right)m.$$

Ahora aplicamos el teorema del coseno en el triángulo ABD:

$$\left(\overline{AB}\right)^2 = 100^2 + \left[90\left(\sqrt{6}-\sqrt{2}\right)\right]^2 - 2\cdot 100\cdot 90\left(\sqrt{6}-\sqrt{2}\right)\cdot cos\,45^{\circ} = 10000 + 8100\left(8-4\sqrt{3}\right) - 9000\left(2\sqrt{3}-2\right) = 92800 - 50400\sqrt{3}$$

$$\overline{AB} = \sqrt{92800 - 50400\sqrt{3}} \simeq 74,2\,m$$