NÚMEROS COMPLEJOS

[1.1] Expresar en forma binómica: $z=\left(1+\sqrt{3}\,i\right)^9+\left(1-\sqrt{3}\,i\right)^9$

Solución:

Teniendo en cuenta que $1+\sqrt{3}i=2_{\pi/3} \quad \wedge \quad 1-\sqrt{3}i=2_{-\pi/3}$

$$z=\left(1+\sqrt{3}i\right)^9+\left(1-\sqrt{3}i\right)^9=\left(2_{\pi/3}\right)^9+\left(2_{-\pi/3}\right)^9=\left(2^9\right)_{3\pi}+\left(2^9\right)_{-3\pi}=$$

$$=2^9(\cos 3\pi+i\,\mathrm{sen}\,3\pi)+2^9(\cos(-3\pi)+i\,\mathrm{sen}(-3\pi))=2^9\cdot 2\cos 3\pi=2^{10}\cdot(-1)=-2^{10}=-1024$$

[1.2] Calcular:

a) $z=\left(1+\sqrt{3}\,i\right)^{\frac{3}{4}}$

b) $z=\dfrac{1+i^{100}}{\left(1+i\right)^{100}}$

c) $z=1+i+\left(1+i\right)^2+\left(1+i\right)^3+\ldots+\left(1+i\right)^{20}$

Solución:

a) $z=\left(1+\sqrt{3}\,i\right)^{\frac{3}{4}}$

$$|z| = \sqrt{1^2+(\sqrt{3})^2} = 2 \quad \wedge \quad \theta = \text{arctg}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

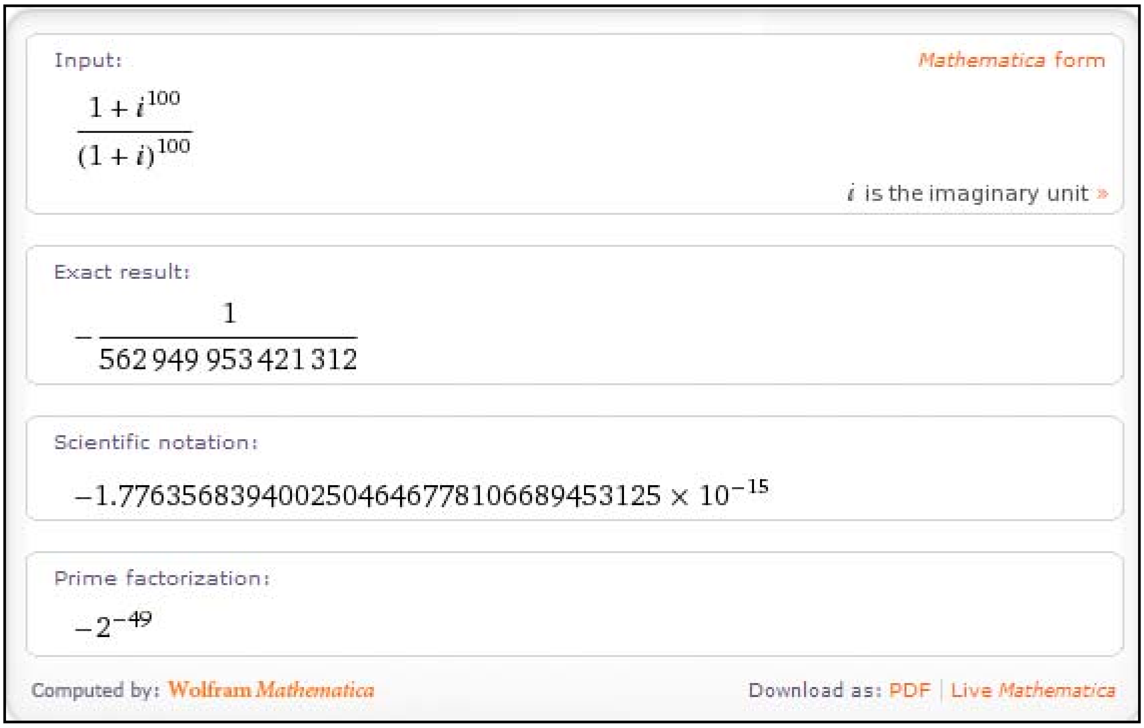
Por tanto:

$$z = \left(1+\sqrt{3}\,i\right)^{\frac{3}{4}} = \sqrt[4]{\left(1+\sqrt{3}i\right)^3} = \sqrt[4]{\left(2_{\frac{\pi}{3}}\right)^3} = \sqrt[4]{8_{\pi}} = \sqrt[4]{8}_{\frac{\pi+2k\pi}{4}} \quad \text{k=0,1,2,3}$$

b) $z = \dfrac{1+i^{100}}{(1+i)^{100}}$

Teniendo en cuenta que $1+i = \sqrt{2}_{\frac{\pi}{4}}$ y que $i^4 = 1$:

$$z = \frac{1+i^{100}}{(1+i)^{100}} = \frac{1+\left(i^4\right)^{25}}{\left(\sqrt{2}_{\frac{\pi}{4}}\right)^{100}} = \frac{1+1}{\left(\sqrt{2}\right)^{100}_{\frac{100\pi}{4}}} = \frac{2}{2^{50}_{25\pi}} = \frac{2}{2^{50}(\cos\ 25\pi + i\ \text{sen}\ 25\pi)} = \frac{2}{2^{50}(-1)} = -2^{-49}$$



c) $z = 1+i+(1+i)^2+(1+i)^3+\ldots+(1+i)^{20}$

Comprobamos que es una progresión geométrica de razón $r = 1+i$, por tanto:

$$z = \frac{a_n r - a_1}{r-1} = \frac{(1+i)^{20}(1+i)-(1+i)}{1+i-1} = \frac{(1+i)^{20}(1+i)-(1+i)}{i} = -i\left[(1+i)^{21}-(1+i)\right]$$

Calculamos en primer lugar $(1+i)^{21}$:

$$(1+i)^{21} = \left(\sqrt{2}_{\frac{\pi}{4}}\right)^{21} = (\sqrt{2})^{21}_{\frac{21\pi}{4}} = 2^{\frac{21}{2}}_{\frac{5\pi}{4}} = 2^{\frac{21}{2}}(\cos\frac{5\pi}{4} + i\,\mathrm{sen}\,\frac{5\pi}{4}) = 2^{\frac{21}{2}}(\frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = 2^{10}(-1-i)$$

Sustituyendo:

$$z = -i\left[2^{10}(-1-i) - (1+i)\right] = i(1+i)\left(2^{10}+1\right) = 1025(-1+i)$$

sum(1+I)^k,k=1 to 20

Result:

$$\sum_{k=1}^{20}(1+i)^k = -1025 + 1025\,i$$

Input: *Mathematica* form

$$(1+i)+(1+i)^2+(1+i)^3+(1+i)^4+(1+i)^5+(1+i)^6+(1+i)^7+(1+i)^8+(1+i)^9+(1+i)^{10}+(1+i)^{11}+(1+i)^{12}+(1+i)^{13}+(1+i)^{14}+(1+i)^{15}+(1+i)^{16}+(1+i)^{17}+(1+i)^{18}+(1+i)^{19}+(1+i)^{20}$$

i is the imaginary unit »

Result:

$-1025 + 1025\,i$

Magnitude:

$1025\sqrt{2} \approx 1449.57$

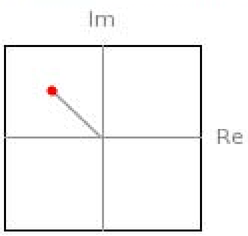
Phase:

$\frac{3\pi}{4} = 135°$

Polar form:

$1025\sqrt{2}\, e^{\frac{1}{4} i (3\pi)}$

Position in the complex plane:



Gaussian prime factorization:

$-(1+i)\times(1+2\,i)^2\times(2+i)^2\times(4+5\,i)\times(5+4\,i)$

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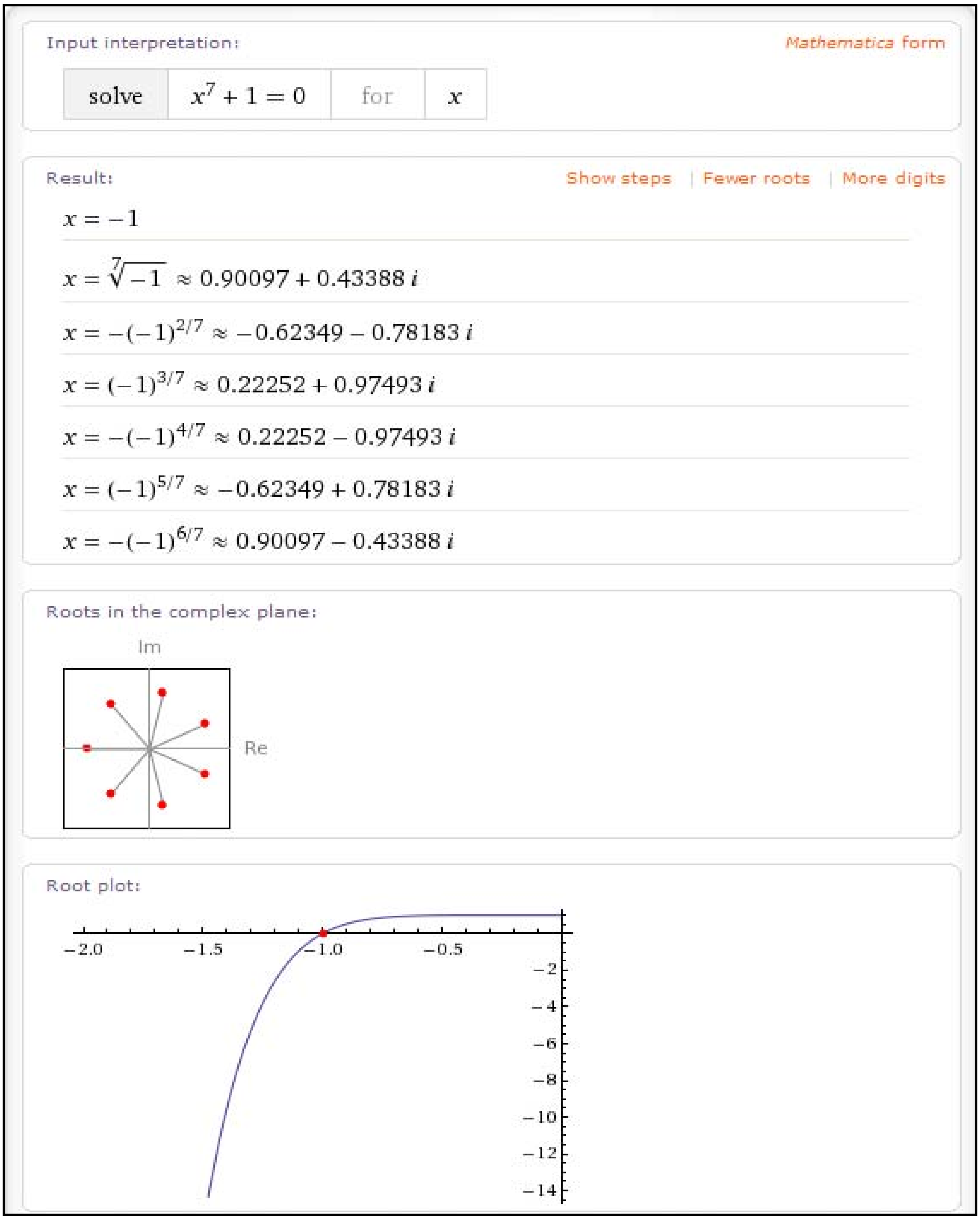
[1.3] Resolver las siguientes ecuaciones algebraicas:

a) $x^7+1=0$

b) $x^7-9x^4+8x=0$

Solución:

1) $x^7+1=0 \quad \Rightarrow \quad x^7=-1 \quad \Rightarrow \quad x=\sqrt[7]{-1}=\sqrt[7]{1_\pi}=\left(1_{\frac{\pi+2k\pi}{7}}\right) \qquad k=0,1,2,3,4,5,6$



2) $x^7 - 9x^4 + 8x = 0 \Rightarrow x\left[x^6 - 9x^3 + 8\right] = 0 \Rightarrow \begin{cases} x = 0 \\ x^6 - 9x^3 + 8 = 0 \end{cases}$

Como $x^6 - 9x^3 + 8 = 0$ es una ecuación bicuadrada, realizamos el cambio $x^3 = t$, obteniendo la ecuación de segundo grado $t^2 - 9t + 8 = 0$, cuyas raíces son $t = 8$ y $t = 1$. Por tanto, obtenemos las ecuaciones $x^3 = 8$ y $x^3 = 1$.

- $x^3 = 8 \Rightarrow x = \sqrt[3]{8_0} = 2_{\frac{0+2k\pi}{3}} \qquad k = 0,1,2$

  Para $k = 0$: $x = 2$

  Para $k = 1$: $x = -1 + \sqrt{3}\,i$

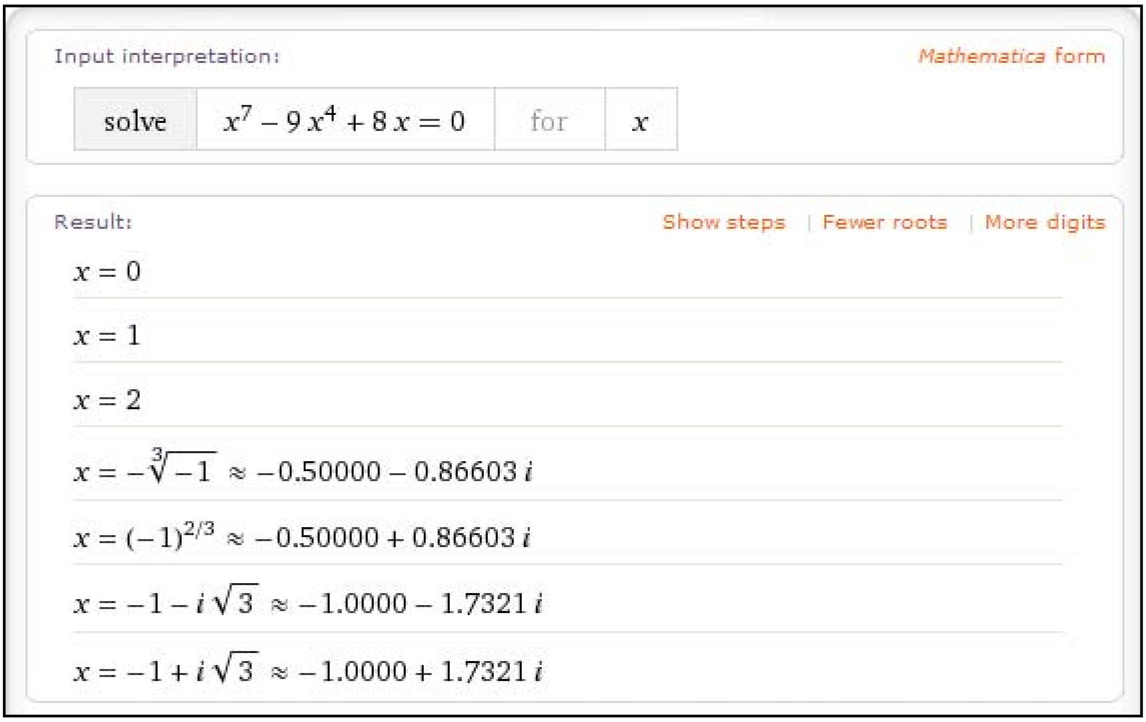
  Para $k = 2$: $x = -1 - \sqrt{3}\,i$

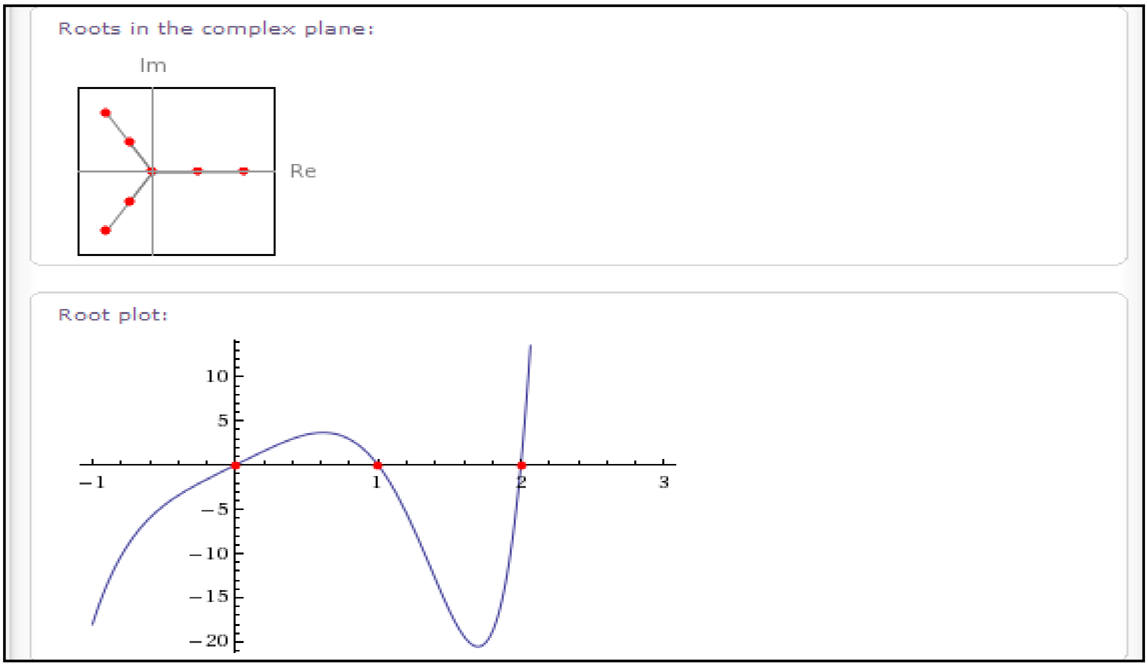
- $x^3 = 1 \Rightarrow x = \sqrt[3]{1_0} = 1_{\frac{0+2k\pi}{3}} \qquad k = 0,1,2$

  Para $k = 0$: $x = 1$

  Para $k = 1$: $x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

  Para $k = 2$: $x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$





[1.4] Hallar los números reales (a) y (b) de forma que el complejo $z=\dfrac{3b-2ai}{4-3i}$, sea un número real y tenga módulo la unidad. Determinar z .

Solución:

$$z=\frac{3b-2ai}{4-3i}\equiv\frac{(3b-2ai)(4+3i)}{25}=\frac{12b+6a+(9b-8a)i}{25}=\frac{12b+6a}{25}+\frac{9b-8a}{25}i$$

Si $z\in\mathbb{R}$: $\quad 9b-8a=0 \;\rightarrow\; a=\dfrac{9b}{8}$

Si $|z|=1 \;\rightarrow\; |z|=\left|\dfrac{12b+6a}{25}\right|=1 \;\rightarrow\; \dfrac{12b+6a}{25}=\pm1 \;\rightarrow\; \begin{cases}12b+\dfrac{54b}{8}=25\\[2mm] 12b+\dfrac{54b}{8}=-25\end{cases} \;\rightarrow$

$$\rightarrow \begin{cases} a=\dfrac{3}{2}, & b=\dfrac{4}{3}\\[2mm] a=-\dfrac{3}{2}, & b=-\dfrac{4}{3}\end{cases}$$

Entonces:

$$\begin{cases} z=\dfrac{12b+6a}{25}=\dfrac{12\cdot\dfrac{4}{3}+6\cdot\dfrac{3}{2}}{25}=\dfrac{16+9}{25}=1\\[4mm] z=\dfrac{12b+6a}{25}=\dfrac{12\cdot\dfrac{-4}{3}+6\cdot\dfrac{-3}{2}}{25}=\dfrac{-16-9}{25}=-1\end{cases}$$

Otra forma de resolver:

$$|z|=1 \;\rightarrow\; z=\pm 1 \;\rightarrow\; \frac{3b-2ai}{4-3i}=\pm 1 \;\rightarrow\; \begin{cases} 3b-2ai=4-3i \;\rightarrow\; \begin{cases} 3b=4 \\ -2a=-3 \end{cases} \rightarrow \begin{cases} b=\dfrac{4}{3} \\ a=\dfrac{3}{2} \end{cases} \\ \\ 3b-2ai=-4+3i \;\rightarrow\; \begin{cases} 3b=-4 \\ -2a=3 \end{cases} \rightarrow \begin{cases} b=-\dfrac{4}{3} \\ a=-\dfrac{3}{2} \end{cases} \end{cases}$$

[1.5] Operando sólo con valores principales, resolver en $\mathbb{C}$ la ecuación:

$$i\left[1-\log_z(z-zi)\right]=1$$

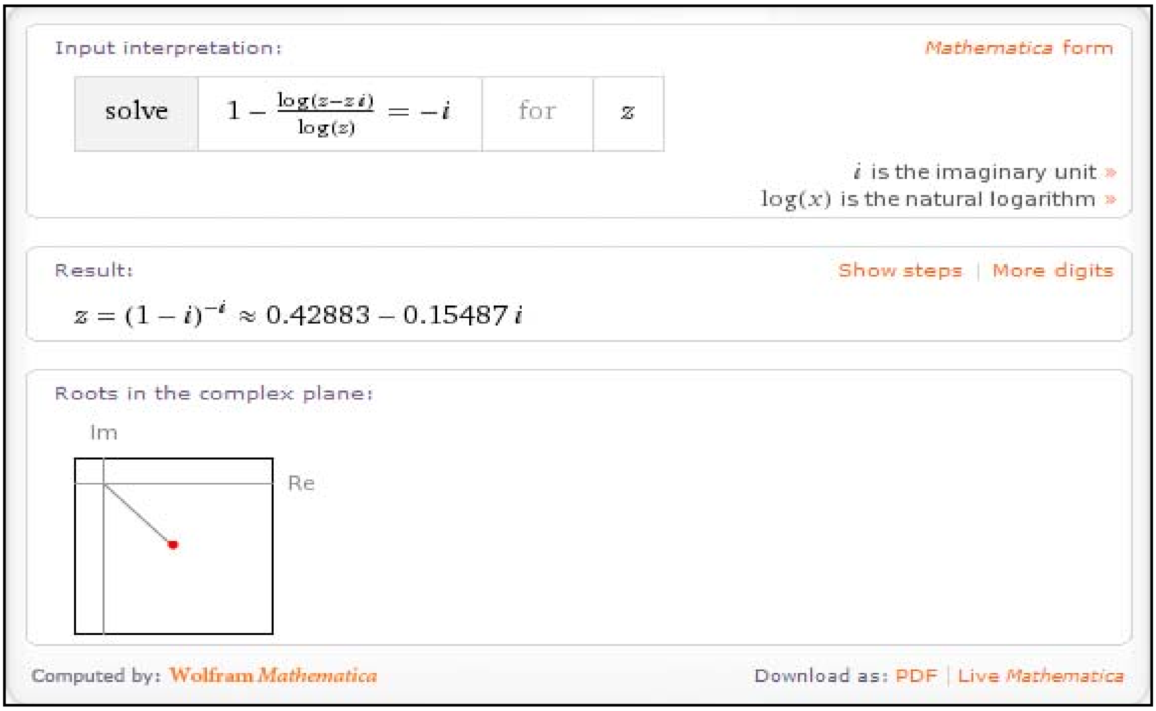
Solución:

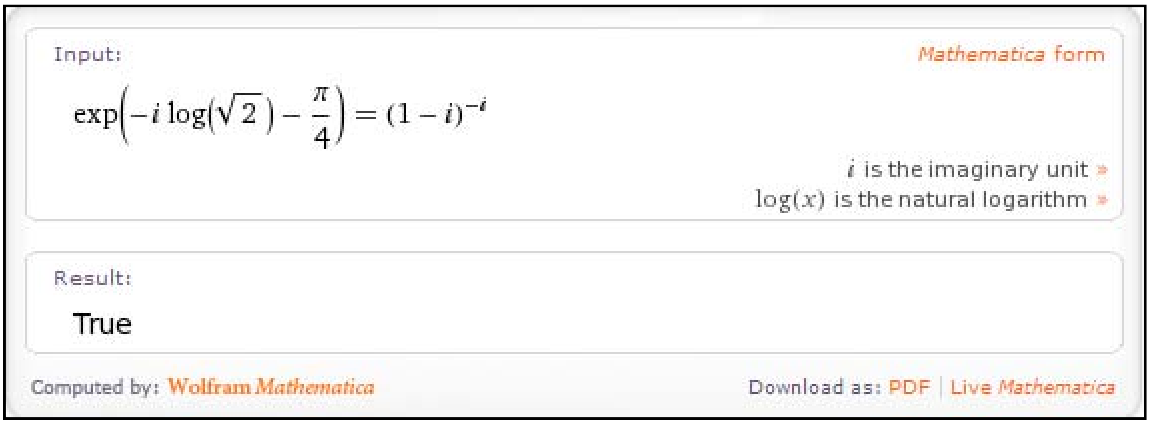
$$i\left[1-\log_z(z-zi)\right]=1 \;\Rightarrow\; 1-\frac{\ln(z-zi)}{\ln z}=-i \;\Rightarrow\; 1+i=\frac{\ln z+\ln(1-i)}{\ln z} \;\Rightarrow$$

$$1+i=1+\frac{\ln(1-i)}{\ln z} \;\Rightarrow\; i=\frac{\ln(1-i)}{\ln z} \;\Rightarrow\; \ln z=\frac{\ln(1-i)}{i} \;\Rightarrow\; \ln z=-i\left[\ln(1-i)\right]$$

$$\Rightarrow\; \ln z=-i\left[\ln(\sqrt{2})_{-\pi/4}\right] \;\Rightarrow\; \ln z=-i\left[\ln(\sqrt{2})+i\left(-\frac{\pi}{4}+2k\pi\right)\right] \;\Rightarrow$$

$$\ln z=-i\left[\ln(\sqrt{2})+i\left(-\frac{\pi}{4}\right)\right] \;\Rightarrow\; \ln z=-i\ln\sqrt{2}-\frac{\pi}{4} \;\Rightarrow\; z=e^{-i\ln\sqrt{2}-\frac{\pi}{4}}$$





[1.6] Resolver la ecuación en números complejos: $z^2-(3+i)z+4=0$. Determinar el logaritmo neperiano de las raíces obtenidas.

Solución:

$$z^2-(3+i)z+4=0 \quad\to\quad z=\frac{3+i\pm\sqrt{(3+i)^2-16}}{2}=\frac{3+i\pm\sqrt{-8+6i}}{2}=$$

$$=\frac{3+i\pm(1+3i)}{2}=\begin{cases}\dfrac{4+4i}{2}=2+2i\\[2mm] \dfrac{2-2i}{2}=1-i\end{cases}$$

En efecto:

$$\sqrt{-8+6i}=a+bi \quad\to\quad -8+6i=(a+bi)^2 \quad\to\quad -8+6i=a^2-b^2+2abi$$

De donde, identificando partes reales real e imaginaria:

$$\begin{cases}-8=a^2-b^2\\ 6=2ab\end{cases} \quad\to\quad b=\frac{3}{a} \quad\wedge\quad -8=a^2-\frac{9}{a^2} \quad\to\quad a^4+8a^2-9=0 \quad\to$$

$$a^2=\frac{-8\pm\sqrt{64+36}}{2}=\frac{-8\pm10}{2}=\begin{cases}1\\ -9\ \ \text{(no es posible)}\end{cases} \quad\to\quad a=\pm1 \quad\wedge\quad b=\pm3$$

El logaritmo neperiano de las raíces obtenidas es:

$$\ln(2+2i)=\ln\left(\sqrt{8}_{\frac{\pi}{4}}\right)=\ln\sqrt{8}+i\left(\frac{\pi}{4}+2k\pi\right)=\frac{3}{2}\ln 2+i\left(\frac{\pi}{4}+2k\pi\right)$$

$$\ln(1-i)=\ln\left(\sqrt{2}_{-\frac{\pi}{4}}\right)=\ln\sqrt{2}+i\left(-\frac{\pi}{4}+2k\pi\right)=\frac{1}{2}\ln 2+i\left(-\frac{\pi}{4}+2k\pi\right)$$

Input interpretation: *Mathematica* form

solve $z^2 - (3+i)\,z + 4 = 0$ for z

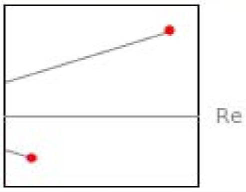
i is the imaginary unit »

Results: Show steps

$z = 1 - i$

$z = 2 + 2\,i$

Roots in the complex plane:



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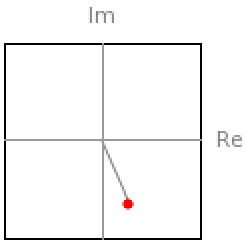
Input: *Mathematica* form

$\log(1 - i)$

i is the imaginary unit »

$\log(x)$ is the natural logarithm »

Position in the complex plane:



Alternate form:

$$\frac{\log(2)}{2} - \frac{i\,\pi}{4}$$

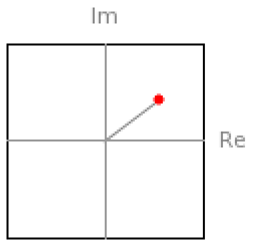
Input: *Mathematica* form

$\log(2 + 2\,i)$

i is the imaginary unit »

$\log(x)$ is the natural logarithm »

Position in the complex plane:



Alternate forms:

$$\frac{3\log(2)}{2}+\frac{i\,\pi}{4}$$

$$\frac{\log(8)}{2}+\frac{i\,\pi}{4}$$

[1.7] Expresar en forma binómica las soluciones de la ecuación:

$$(3-2i)(\rho e^{i\theta})^2-6i-4=0$$

Solución:

Recuérdese que $\rho e^{i\theta}=\rho(\cos\theta+i\,\mathrm{sen}\,\theta)=\rho_\theta=z$.

Luego se trata de resolver la ecuación: $(3-2i)z^2-(4+6i)=0$

$$z^2=\frac{4+6i}{3-2i}=\frac{(4+6i)(3+2i)}{3^2+(-2)^2}=\frac{12+18i+8i-12}{13}=\frac{26i}{13}=2i$$

$$z^2=2i \quad\Rightarrow\quad z=\sqrt{2i}=\sqrt{2_{\pi/2}}=\left(\sqrt{2}\right)_{\frac{\pi/2+2k\pi}{2}} \quad (k=0,1)$$

$$z=\begin{cases}(\sqrt{2})_{\pi/4}=\sqrt{2}\left(\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right)=1+i\\[2ex](\sqrt{2})_{5\pi/4}=\sqrt{2}\left(-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right)=-1-i\end{cases}$$

Otra forma de resolver (aplicando la definición de raíz cuadrada):

$$\sqrt{2i}=x+yi \quad (x,y\in\mathbb{R}) \quad\Rightarrow\quad 2i=(x+yi)^2=x^2-y^2+2xyi \quad\Leftrightarrow$$

$$\Leftrightarrow\quad\begin{cases}x^2-y^2=0\\ 2xy=2\end{cases}\quad\Rightarrow\quad\begin{cases}y=\dfrac{1}{x}\\[1ex] x^2-\dfrac{1}{x^2}=0\end{cases}\quad\Rightarrow\begin{cases}y=\dfrac{1}{x}\\[1ex] x^4-1=0\end{cases}\Rightarrow\quad x=\pm1 \quad (x\in\mathbb{R}) \quad y=\pm1$$

Input: Mathematica form

$(3-2i)z^2-4-6i=0$

i is the imaginary unit »

Result:

$(3-2i)z^2-(4+6i)=0$

Alternate forms:

$z^2-2i=0$

$(3-2i)(z-(1+i))(z+(1+i))=0$

Alternate form assuming all variables are real:

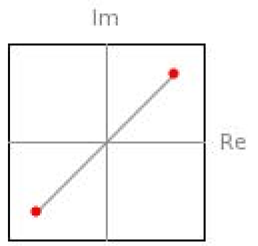
$3z^2+i(-2z^2-6)-4=0$

Complex solutions:

$z=-1-i$

$z=1+i$

Roots in the complex plane:



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[1.8] Describir el lugar geométrico de los puntos $z \in \mathbb{C}$, que verifican la condición:

$$\left|\frac{z-2}{z+2}\right|=3$$

Solución:

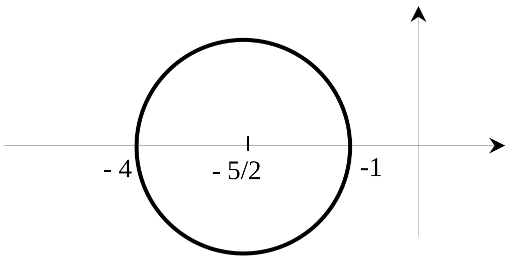
Siendo $z=x+yi$ se tiene:

$$\left|\frac{z-2}{z+2}\right|=3 \quad \Rightarrow \quad \left|\frac{(x-2)+yi}{(x+2)+yi}\right|=3 \quad \Rightarrow \quad \sqrt{(x-2)^2+y^2}=3\sqrt{(x+2)^2+y^2} \quad \Rightarrow$$

$$(x-2)^2+y^2=9[(x+2)^2+y^2] \quad \Rightarrow \quad 8x^2+40x+32+8y^2=0 \quad \Rightarrow$$

$$x^2+5x+4+y^2=0 \quad \Rightarrow \quad \left(x+\frac{5}{2}\right)^2+y^2=\left(\frac{3}{2}\right)^2$$

Circunferencia de centro $C=(-5/2,0)$ y radio $r=3/2$

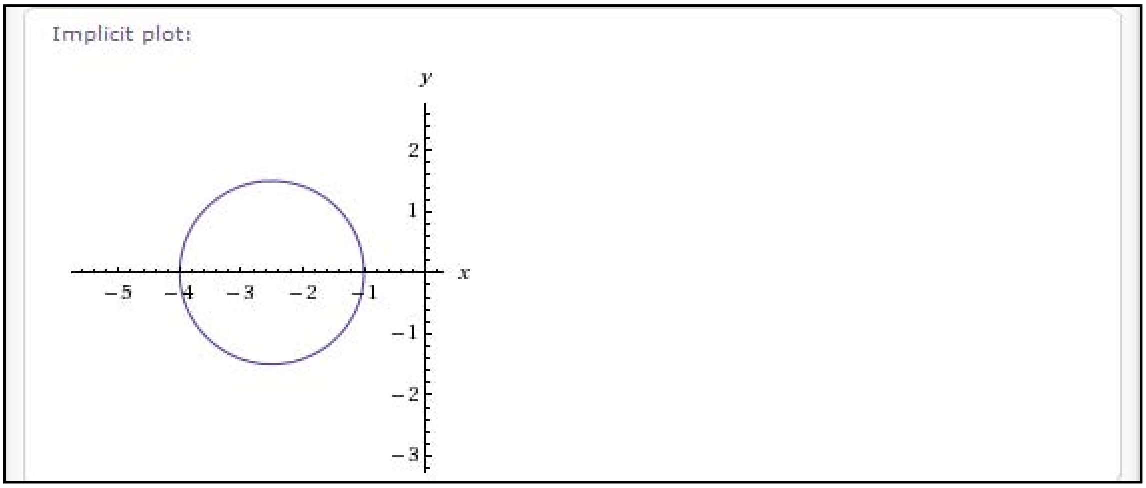


Input interpretation:

simplify	$\sqrt{(x-2)^2+y^2} = 3\sqrt{(x+2)^2+y^2}$

Result:

$x(x+5)+y^2+4=0$



[1.9] Determinar la ecuación del lugar geométrico de los $z=x+yi$ del plano complejo tales que la razón de las distancias de z a los puntos 1 y -1 tienen el valor constante k

Solución:

$$\frac{|z-1|}{|z+1|}=k \quad (k>0) \quad \Rightarrow \quad \frac{(x-1)^2+y^2}{(x+1)^2+y^2}=k^2 \quad \Rightarrow \quad x^2-2x+1+y^2=k^2(x^2+2x+1+y^2)$$

$$x^2(k^2-1)+y^2(k^2-1)+2x(k^2+1)+k^2-1=0$$

- Si $k \neq 1$:

$$x^2+y^2+2x\frac{k^2+1}{k^2-1}+1=0 \quad \Rightarrow \quad \left(x+\frac{k^2+1}{k^2-1}\right)^2+y^2=\left(\frac{k^2+1}{k^2-1}\right)^2-1=\frac{4k^2}{(k^2-1)^2}$$

Se trata de una <u>circunferencia</u>, cuyo centro y radio son: $C\left(-\frac{k^2+1}{k^2-1},0\right) \qquad r=\frac{2k}{|k^2-1|}$

- Si $k=1$: $x=0$ (<u>eje de ordenadas</u>)

[1.10] Los puntos $(1,5)$ y $(1,7)$ son vértices opuestos de un octógono regular. Determinar los restantes vértices.

Solución:

Si $(1,5)$ y $(1,7)$ son vértices opuestos, el centro del octógono regular está en su semisuma, es decir, en el punto $(1,6)$, y el radio vector es $(1,7)-(1,6)=(0,1)$. Girando este radio vector $2\pi/8$ radianes sucesivamente, se obtienen los restantes vértices.

Utilizando números complejos se tiene:

$$V_1=(1,7)$$

$$V_2=(1,6)+(0,1)\cdot 1_{\pi/4}=(1,6)+(0,1)\cdot(\sqrt{2}/2,\sqrt{2}/2)=(1,6)+(-\sqrt{2}/2,\sqrt{2}/2)=$$
$$=(1-\sqrt{2}/2,6+\sqrt{2}/2)$$

$$V_3=(1,6)+(-\sqrt{2}/2,\sqrt{2}/2)\cdot 1_{\pi/4}=(1,6)+(-\sqrt{2}/2,\sqrt{2}/2)\cdot(\sqrt{2}/2,\sqrt{2}/2)=(1,6)+(-1,0)=$$
$$=(0,6)$$

$$V_4=(1,6)+(-1,0)\cdot 1_{\pi/4}=(1,6)+(-1,0)\cdot(\sqrt{2}/2,\sqrt{2}/2)=(1,6)+(-\sqrt{2}/2,-\sqrt{2}/2)=$$
$$=(1-\sqrt{2}/2,6-\sqrt{2}/2)$$

Haciendo uso de la simetría de los vértices respecto del centro del octógono resulta:

$$V_5=(1,6)-(0,1)=(1,5)$$

$$V_6=(1,6)-(-\sqrt{2}/2,\sqrt{2}/2)=(1+\sqrt{2}/2,6-\sqrt{2}/2)$$

$V_7 = (1,6) - (-1,0) = (2,6)$

$V_8 = (1,6) - (-\sqrt{2}/2\,, -\sqrt{2}/2) = (1+\sqrt{2}/2\,, 6+\sqrt{2}/2)$